On the stability problem in the $O(N)$ nonlinear sigma model.

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Abstract

The stability problem for the $O(N)$ nonlinear sigma model in the $2 + \epsilon$ dimensions is considered. We present the results of the $1/N^2$ order calculations of the critical exponents (in the $2 < d < 4$ dimensions) of the composite operators relevant for this problem. The arguments in the favor of the scenario with the conventional fixed point are given.

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In a recent time the possibility of a scenario with the nonconventional fixed points in $2+\epsilon$ expansions for various models was widely discussed. It has been firstly observed by Kravtsov, Lerner and Yudson in Q matrix model [1], and proved later by Wegner for N vector, unitary and orthogonal matrix models [2–4], that a certain class of composite operators with $2s$ fields and gradients endangers the stability of nontrivial fixed points in $2 + \epsilon$ expansions for these models. In the following we restrict ourselves to the case of the N vector model. The one - loop critical exponents of the corresponding operators in this model read as [2]

$$x_s = 2s - \epsilon \frac{s(s-1)}{N-2} + O(\epsilon^2). \tag{1}$$

Judging from these results one could argue that for a sufficiently large s operators become relevant (operators with $y > 0$, $y = 0$ and $y < 0$ ($y \equiv d - x$) are relevant, marginal and irrelevant, resp.) and the conventional nontrivial fixed point becomes unstable against an infinite number of high–gradient perturbations. This would have important consequences for the present understanding of many problems that rely on $2 + \epsilon$ expansions.

To get a more detailed notion of this problem it seems a reasonable to take advantages of the $1/N$ expansions, which being nonperturbative in its nature relates the $2+\epsilon$ and $4-\epsilon$ expansions. The latter is commonly believed to be free of any pathologies. Moreover, in this approach the corresponding operators look extremely simple — σ^s, where σ is the auxiliary field. The critical exponents for this set of operators had been calculated in Ref. [5], and independently in Ref. [6]:

$$x_s = 2s + \frac{\eta_1}{N}s(d-1)\frac{(s-1)d(d-3) - 2(d-2)}{4-d}, \tag{2}$$

where $\eta_1 = 4(2-\mu)\Gamma(2\mu-2)/\Gamma^2(\mu-1)\Gamma(2-\mu)\Gamma(\mu+1)$ and $\mu \equiv d/2$. One can see that the problem similar to those in $2 + \epsilon$ expansion arises below $d = 3$.

The next attempt to attack the stability problem has been undertaken in the papers [7,8], where the critical exponents x_s have been calculated with the ϵ^2 accuracy. However, the full answer for x_s given in Refs. [7,8] is not consistent with those (Eq. (2)) obtained in Refs. [5,6] in the $1/N$ approach, and with the expression for index ω ($\omega = x_2 - d$), recently obtained in Ref. [11] with $1/N^2$ accuracy.

The aim of the present paper is the calculation of the critical exponents x_s with $1/N^2$ accuracy. This provides us the deeper insight into problem and allow to suggest the realistic scenario for its solution. The computation scheme developed below is of own interest. We remind that only few critical indices are know beyond the $1/N$ order [9–12]. All of them had been calculated by the specific methods, which are not applicable in a general situation. With this remark we postpone the discussion to the end of the paper and proceed to the calculations.

The renormalized action for the nonlinear sigma model can be written as [13]:

$$S = -\frac{Z_1}{2}(\partial\phi)^2 - \frac{1}{2}\sigma K_\Delta M^{-2\Delta}\sigma + \frac{Z_2}{2}\sigma\phi^2 + \frac{1}{2}\sigma K\sigma. \tag{3}$$

Here ϕ^A is the vector field ($A = 1, \ldots, N$), σ is the auxiliary scalar field; Δ and M are regularization parameter and mass, respectively. The kernel K is determined from the

requirement of the cancellation of the self–energy diagram contribution of the sigma field: $K(x) = -N/2G_\phi^2(x)$. $G_\phi(x)$ is the propagator of the ϕ^A field ($< \phi^A(x)\phi^B(0) > = \delta^{AB}G_\phi(x)$). The regularized kernel K_Δ is defined as $K_\Delta(x) = K(x)x^{-2\Delta}$. The divergencies appearing at the calculations of the Feynman diagrams as poles in Δ are removed by the suitable choice of the constants Z_1 and Z_2. Henceforth we will use the minimal subtraction scheme (MS).

Unfortunately, the model under consideration is not the multiplicatively renormalized [13]. This means that the freedom in the choice of the renormalization prescriptions can not be compensated by the redefinition of the fields and the parameters of Lagrangian. Thus one fails to apply the standard RG methods for the calculations of the critical indices. The general discussion of this topic can be found in Refs. [13,5]. Here we remind only one basic property important for the following, namely, the correlation functions of the fields and composite operators in the model under consideration are scale invariant. There is, the renormalized one particle irreducible Green's function $\Gamma^{O,n}(p, p_i) = \langle O(p)\Phi(p_1)...\Phi(p_n)\rangle$ with operator insertion ($\Phi \equiv \phi, \sigma$) satisfies the equation $\Gamma^{O,n}(\lambda p, \lambda p_i) = \lambda^{x_O - n_\Phi x_\Phi}\Gamma^{O,n}(p, p_i)$. As usual, $n_\Phi x_\Phi$ means $n_\phi x_\phi + n_\sigma x_\sigma$ and x_O, x_Φ are the critical dimensions of operator O and fields Φ, respectively. In the following the composite operators and Green functions are assumed to be renormalized, i.e. the all needed counterterms are taking into account.

The effective algorithm for the computation of the anomalous dimensions of composite operators in the first order of $1/N$ expansion exploiting the property of scale invariance has been developed in the paper of Vasil'ev and Stepanenko [5]. They have shown that the simple correlation between the Δ pole residues of the Green's functions and the corresponding anomalous dimensions exists. The generalization of this method to all order of $1/N$ expansion (VS scheme) is given below. The basic formula we used for the calculation of the anomalous dimensions reads:

$$u\Gamma^{O,n}(p, p_i, M) = \lim_{\Delta \to 0} 2\Delta \sum_{\{G\}} n_\sigma^G \Gamma_G^{O,n}(p, p_i, M, \Delta). \tag{4}$$

Here $\Gamma^{O,n}(p, p_i, M)$ is the $1PI$ n – point Green's function with operator insertion. A operator O is assumed to be the operator with dimension (scaling one). The sum runs over whole set of the diagrams (including those with counterterms); n_σ^G is the number of the sigma lines in the diagram G; $u = -\gamma_O + n_\Phi\gamma_\Phi$ (γ is used for anomalous dimensions).

To derive the Eq. (4) we note that the scale invariance results in the following form of $\Gamma^{O,n}(p, p_i, M)$:

$$\Gamma^{O,n}(p, p_i, M) \equiv \lim_{\Delta \to 0} \sum_{\{G\}} \Gamma_G^{O,n}(p, p_i, M, \Delta) = p^U (M/p)^u \widetilde{\Gamma}(p_i/p) \tag{5}$$

where $U = x_O^{can} - n_\Phi x_\Phi^{can}$. Then acting by $M\partial_M$ on $\Gamma^{O,n}(p, p_i, M)$ and taking into account that the only dependence on M in the diagrams results from the propagator of σ field ($G_\sigma = M^{2\Delta}K_\Delta^{-1}$) (we remind that all counterterms are chosen independent on M (MS scheme)) one immediately obtains the Eq. (4). Note, the finiteness of the lhs of Eq. (4) ensures the cancellation of all Δ poles in the rhs except for the first order ones.

Further, in the case of operators $\{O_i\}$ mixing under renormalization one should seek for the proper scaling operators as the linear combinations: $\tilde{O}_i = \sum c_{ik}O_k$. To determine both the anomalous dimensions and the form of those one should consider the Eq. (4) for n – point Green's functions with insertion of operators $\tilde{O}_i$ for different n.

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We apply now the above scheme to the calculation of the critical exponents of σ^s operators in $1/N^2$ order. In spite of the fact that there are a lot of operators with the same canonical dimension, which could have admixed to σ^s, this does not happen in the first order of $1/N$ expansion [5]. The renormalized σ^s operator reads as $[\sigma^s] = Z_s \sigma^s$ ($Z_s = 1 + q_s/N\Delta$) and is the proper scaling operator in this order. (Henceforth we use the standard notation $[O]$ for the renormalized operator.) Going to the next order one finds the counterterms of the following form are required – $(\sigma^s, \sigma^{s-2}\partial^2\sigma, \partial^2\sigma^{s-1})$, while all diagrams describing $\sigma^2 \to \partial^4$ transition have not the divergencies. This force us to conclude that in $1/N^2$ order the scaling operators under consideration have form

$$[O_s] = [\sigma^s] + \frac{\alpha_1}{N}[\sigma^{s-2}\partial^2\sigma] + \frac{\alpha_2}{N}[\partial^2\sigma^{s-1}] + O(\frac{1}{N^2}). \tag{6}$$

To determine $\alpha_{1,2}$ (really, we need α_1 only) one must consider Eq. (4) for $(s-1)$ point Green's function of sigma fields with insertion of the operator $[O_s]$. Doing the same for s – point function $\Gamma^{O_s,s}(p, p_i)$ one obtains the anomalous dimension of the operator in hand.

However, there is the much simple and elegant way to solve the mixing problem. Let us remind that the nonlinear σ model is the simplest example of the conformal field theory (CFT) ($d > 2$) [6,14,15]. Following along the lines of the paper [16] we suppose that the conformal operators (CO's) and their total derivatives form a complete basis in the space of all operators. In this case any exact scaling operator not being a total derivative of other is a conformal one, the opposite is evidently true. (Of course, the more accurate statement needs when there is a degeneracy of critical dimensions.) This observation being combined with other the well known fact – vanishing of two – point correlator of CO's with different scaling dimensions [16] – considerably simplifies the solution of the mixing problem. For the conciseness we illustrate this idea on the concrete examples, the generalization being straightforward.

The only nonderivative scaling (and hence conformal) operators on the levels $s = 1, 2$ (i.e. with the canonical dimensions equal to $2s$) have form $[O_1] = [\sigma]$ and $[O_2] = [\sigma^2] + \alpha(N)\partial^2\sigma$. From the requirement of "orthogonality" $\langle O_2(x)O_1(y)\rangle = 0$ one immediately obtains $\langle [\sigma^2(p)]\sigma(-p)\rangle_{1PI} = \alpha(N)p^2$, with $\alpha(N) = (\eta_1/4N)(\mu-1)(2\mu-3)/(3-\mu)+O(1/N^2)$. The above equality holds in all order of $1/N$ expansion. Note, namely the absence of the logarithmic corrections to the above correlator results in the multiplicative renormalization of σ^s operators in the first $1/N$ order [5].

On the level $s = 3$ two new conformal operators come into a game. On the classical level they reads as $O_3^{(1)} = \sigma^3$ and $O_3^{(2)} = \sigma\partial^2\sigma - \frac{(3-\mu)}{2(5-\mu)}\partial^2\sigma^2$. Then the exact CO's can be written as:

$$[O_3^{c,(1)}] = [O_3^{(1)}] + a_1[O_3^{(2)}] + a_2\partial^2[O_2] + a_3\partial^4[O_1] \tag{7a}$$

$$[O_3^{c,(2)}] = [O_3^{(2)}] + b_1[O_3^{(1)}] + b_2\partial^2[O_2] + b_3\partial^4[O_1] \tag{7b}$$

Here a_i, b_i are some functions of N and μ. From the "orthogonality" of $[O_3^{c,(1,2)}]$ to $[O_2], [O_1]$ one easy finds that $a_2 \sim b_2 \sim b_3 \sim O(1/N)$ and $a_3 \sim O(1/N^2)$. In the same time condition $\langle [O_3^{c,(1)}](x)[O_3^{c,(2)}](y)\rangle = 0$ gives the equation entangling b_1 and a_1, ($a_1 = a_1^0/N + O(1/N^2)$ and $b_1 = b_1^0 + O(1/N)$):

$$\langle [O_3^{(1)}](x)[O_3^{(2)}](y)\rangle + b_1\langle [O_3^{(1)}](x)[O_3^{(1)}](y)\rangle + a_1\langle [O_3^{(2)}](x)[O_3^{(2)}](y)\rangle = O(N^{-4}) \tag{8}$$

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To determine the coefficient b_1^0 one must use VS scheme described above (see also Ref. [5]): $b_1^0 = 2\gamma_{21}/(\gamma_{O_3^{(1)}} - \gamma_{O_3^{(2)}})$; where $\gamma_{O_3^i}$ are the anomalous dimensions of the corresponding operators in the $1/N$ order: (All answers it will be given in the units (η_1/N) and $(\eta_1/N)^2$ for the first and second order of $1/N$ expansion, respectively.)

$$\gamma_{O_3^{(1)}} = 6(2\mu - 1)[\mu(2\mu - 3) - (\mu - 1)]/(2 - \mu)$$
$$\gamma_{O_3^{(2)}} = (2\mu - 1)\{(2\mu - 3)[27\mu - 3(11\mu + 4) + \mu^2 + 9\mu + 2] - 12(\mu - 1)\}/3(2 - \mu). \tag{9}$$

The coefficient γ_{21} arises from the mixing of the operators $O_3^{(1)}$ and $O_3^{(2)}$ in $1/N$ order:

$$\gamma_{21} = \lim_{\Delta \to 0} \Delta \sum_{\{G\}} n^\sigma \Gamma_G^{\sigma^3, 2}(p, p_1, p_2) = 2\mu(2\mu - 3)(4\mu^2 - 1)/3(2 - \mu) + O(1/N^2). \tag{10}$$

Since b_1^0 is known, a_1^0 can be easy obtains from the orthogonality condition (8). One need calculate two correlators $\langle [O_3^{(1(2))}](x)[O_3^{(1(2))}](y)\rangle$ in the leading order and one $\langle [O_3^{(1)}](x)[O_3^{(2)}(y)]\rangle$ in the next to leading order. Note, for the determination of the same coefficient in VS scheme one need to calculate 20 diagrams in $1/N^2$ order.

Obviously, the same scheme can be applied for the determination of coefficient α_1 in the operator $[O_s]$ (see Eq. 6). However, for the persuasiveness we have carried out calculations in the both approaches, which result in the same answer $\alpha_1 = \eta_1 s(s - 1)\bar{\alpha}_1$:

$$\bar{\alpha}_1 = \frac{(2\mu - 3)(\mu - 1)}{4(3 - \mu)} \frac{4s(\mu - 3) + \mu^2 - 5\mu + 12}{4s(2\mu - 3) - \mu^2 - 7\mu + 12}.$$

Thus this trick allows to fix the form of the scaling operator avoiding the cumbersome $1/N^2$ order calculations. (The coefficient α_1 is singular at $\mu = \mu_s \simeq 3/2(1 + 1/16(s - 1))$. But this fact has the simple explanation – at this point the degeneration of $1/N$ order anomalous dimensions occurs (see e.g. formula for b_1^0). So it is only a artifact of used approach, which has to be modified in this case.)

Since the coefficient α_1 is known the anomalous dimension of $[O_s]$ can be determined from Eq. (4) for $\Gamma_s(p, p_i) = \langle [O_s](p)\sigma(p_1)\dots\sigma(p_s)\rangle$. It is instructive to check that the *lhs* and *rhs* of Eq. (4) have the same momentum dependence. This can be done on the formula level and leads us to very nice formula for $u_s^{(2)}$ ($u_s^{(2)} = (-\gamma_s^{(2)} + s\gamma_\sigma^{(2)})$) having obvious resemblance with its counterpart in dimensional regularization scheme:

$$u_s^{(2)} = 2\Delta \sum_G n_\sigma^G [KR'G]_\Delta. \tag{11}$$

Here the sum runs over all diagrams; the KR' operation is the standard operation of the subtractions on the divergent subgraphs; $[..]_\Delta$ means that only first order poles in Δ should be picked out; n_σ^G is the number of σ lines in a diagram G. It is convenient to represent $u_s^{(2)}$ as:

$$u_s^{(2)} = s(s - 1)u_2^{(2)}/2 + s(s - 1)(s - 2)\left\{r_3^{(2)} + 2\bar{\alpha}_1\gamma_{21}\right\}. \tag{12}$$

Here the first two terms arises from the diagrams describing transition $\sigma^2 \to \sigma\sigma$ ($u_2^{(2)}$), and $\sigma^3 \to \sigma\sigma\sigma$ ($r_3^{(2)}$) in the correlator $\Gamma^{\sigma^s, s}(p, p_i)$, while the last one is due to the admixture in

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$[O_s]$. Since the anomalous dimension of σ field (γ_σ) and those of σ^2 operator $(\gamma_2 = 2\gamma_\sigma - u_2)$ are known up to $1/N^2$ order [9,11,12], the problem reduces to the determination of $r_3^{(2)}$. There are 26 diagrams (three 3-loop, ten 4-loop, ten 5-loop, and three 6-loop) which contribute to $r_3^{(2)}$. They all can be calculated with the help of technique developed in [9,10]. The final expression for $r_3^{(2)}$ reads:

$$r_3^{(2)} = \frac{\mu^2(\mu-1)(15\mu^3 - 51\mu^2 + 52\mu - 12)}{2(\mu-2)^2}C +$$
$$\frac{\mu(2\mu-3)(104\mu^5 - 383\mu^4 + 542\mu^3 - 383\mu^2 + 104\mu - 8)}{12(\mu-1)(\mu-2)^2} \tag{13}$$

Here $C = \psi'(1) - \psi'(\mu-1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$. We do not give the complete expression for the critical exponents x_s because of its large size. It is more interesting to look on the corresponding expansions near 2 and 4 dimensions (we write down only leading in s terms):

$$x_s^{2+\epsilon} = 2s - s(s\epsilon/N) - s/2(s\epsilon/N)^2 + .. \tag{14a}$$
$$x_s^{4-\epsilon} = 2s + 6s(s\epsilon/N) - 34s(s\epsilon/N)^2 + .. \tag{14b}$$

It should be stressed that the coefficient at the last term in Eq. (14a) differs from those obtained in the paper [7]. In the $4 - \epsilon$ expansion the exponents x_s (corresponding to $(\phi^2)^s$ operators) has been so far known with ϵ accuracy only. For the additional check of our results we have calculated x_s up to ϵ^2 order:

$$x_s = 2s + 6\epsilon\frac{s(s-2)}{N+8} - \epsilon^2 s \left\{(s-1)[34(s-2)(N+8)+ \right.$$
$$(11N^2 + 92N + 212)] - (13N+44)(N+2)/2\} /(N+8)^3, \tag{15}$$

the above expression being in the full agreement with those obtained in the frame of $1/N$ expansion.

Let us now discuss the obtained results. One can see from the Eqs. (14) that as in the $2+\epsilon$ as in the $4-\epsilon$ expansions of the exponents x_s the second order terms have negative ("wrong") sign. Thus for a large $(s\epsilon/N)$ the critical exponents given by the truncated series (14) become a negative. This might serve a starting point for the speculations on the stability of the conventional fixed points. But, do the first order terms (14) give a well approximation for x_s when $(s\epsilon/N)$ is large? The answer, of course, is negative, since the estimative terms in (14) are $O((s\epsilon/N)^3)$ order. Moreover, the pure combinatorical analysis shows that the k-th order term in the series (14) behaves as s^{k+1} at large s. Thus to obtain the answer for x_s which were sensible for large $s\epsilon/N$ one need sum up all order corrections (see for further discussion Refs. [17,18]). Even taking into account the recent progress in the higher order calculations [12] the feasibility of this program causes the great doubts.

The said above concerns both the $2+\epsilon$, $4-\epsilon$ expansions and the $1/N$ expansion. However, in the case when only a few first terms are calculable, the $1/N$ expansion has the definite advantage in comparison with ϵ ones. Indeed, $1/N$ expansion is more informative, because it contains information on critical exponents in whole interval $2 < \mu < 4$. In the case under

consideration, one can, with some extent of a confidence, judge about the general tendency from the first order results.

Let the function $A_k(\mu)$ is the coefficient at the leading in s term (s^{k+1}) of the k order term in the $1/N$ expansion of the exponent x_s. The function $A_1(\mu)$ and $A_2(\mu)$ are drawn in Fig. 1. It is natural to consider the intervals I_k on which these functions are negative, i.e. if $\mu \in I_k$ then $A_i \leq 0$ for $i = 1, \ldots, k$. (Indeed, it might be said that the stability problem in the $2 + \epsilon$ expansion arises from the negativity of the first two coefficients in the expansion (14a). If one of them were positive, the problem would hardly be considered as serious one.) It is seen that the interval I_2 is considerably smaller than I_1 ($I_1 = [1, \, 1.5]$, while $I_2 \simeq [1, \, 1.2]$). Obviously, if this tendency ($I_k \to 0$ at $k \to \infty$) will hold in the higher order, the stability problem lost its sharpness. Indeed, in this case for any $\epsilon > 0$ the $1/N$ corrections starting from some order will shift the critical dimensions in the true – irrelevant – direction (at least they will be sign - varied).

We conclude with the following remarks: In all available expansions of the N vector model ($2 + \epsilon$, $1/N$, $4 - \epsilon$) there exists the class of operators which acquire a "large" anomalous dimensions in the first orders of the corresponding expansions. This fact is not related to the expansion used, but has the combinatorical origin. The stability problem arises when one tries to extrapolate the first orders results out of their range of applicability. Though we do not state the $2 + \epsilon$ expansion is free of the problems [19,20] it seems they are not related to the high – gradient operators.

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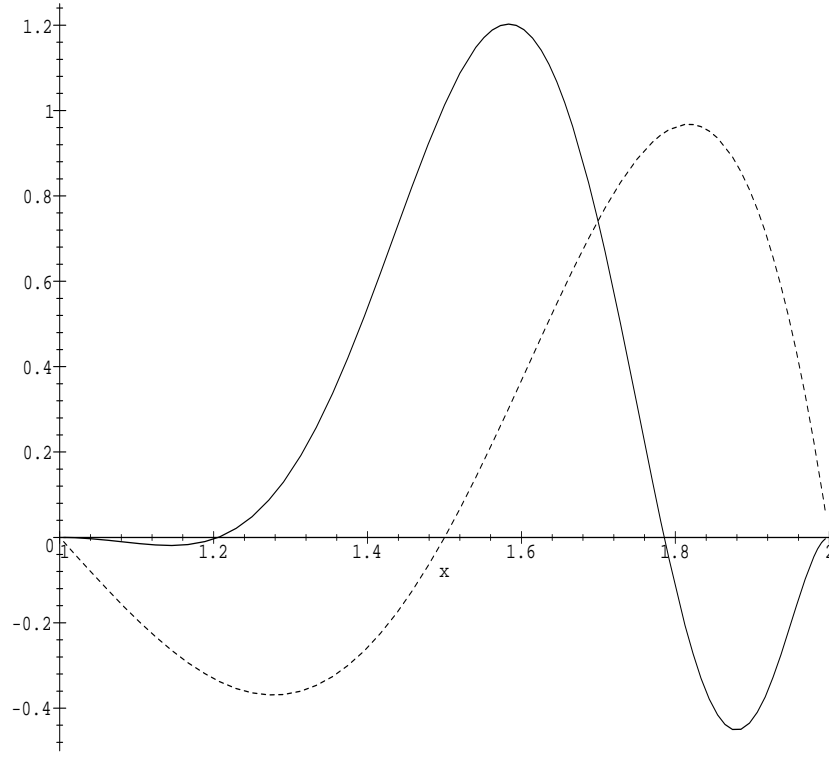
FIGURES

FIG. 1. The functions $A_1(x)$ and $A_2(x)$ are plotted as the functions of the space dimension. The dot line corresponds to $A_1(x)$ (the first order in $1/N$), and the solid line — to $A_2(x)$ (the second order in $1/N$).